LIMIT THEOREMS FOR RAREFACTION OF SET
OF DIFFUSION PROCESSES BY BOUNDARIES.[1]

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ABSTRACT. This paper is devoted to the study of the following problem. We have set of diffusion processes with absorption on boundaries in some region at initial time $t = 0$. It is required to estimate of number of the unabsorbed processes for the fixed time $\tau > 0$. The number of initial processes is considered as function of τ and $\tau \to \infty$.

Consider the N independent diffusion processes which start from different points $x_k \in Q$, $k = \overline{1, N}$ in the initial moment of time $t = 0$. The region $Q \subset R^d$ is open connected region and it is limited by smooth surface ∂Q. All processes are diffusion processes with absorption on the boundary ∂Q. These processes are solutions of the following stochastic differential equations in Q with absorption

$$d\xi(t) = a(t, \xi(t))dt + \sum_{i=1}^{d} b_i(t, \xi(t))dw_i^{(k)}(t) \quad \xi(t) \in R^d \tag{1}$$

$$b_i(t, x), \ a(t, x) : R_+ \times R^d \to R^d.$$

with an initial condition: $\xi(0) = \gamma_k \in Q$.

Here the $W^{(k)}(t) = (w_i^{(k)}(t), \quad 1 \le i \le d)$, $1 \le k \le N$ are independent in totality d-dimensional Wiener processes.

Thus, these processes have the identical diffusion matrices and shift vectors , but they have different initial states.

We will be interested by distribution of number of processes yet not absorbed by boundary ∂Q in the moment of time τ.

This task was offered in [1] as the mathematical model of cleaning of gas from particles (dust, microbes and ect.). Cleaning consists in pass of gas with speed v through the pipe of length l and with permanent section $Q \cup \partial Q$. Walls of pipe absorb the particles. The estimation of efficiency of cleaning can be reduced to the solution of this problem at $\tau = l/v$. In supposition, that particles evolve independently from each other and their movement can be presented as the solutions of (1).

We make the following assumptions: a number and positions of particles are defined by a given determinate limited measure $N(B, \tau)$ at the initial time. Thus $N(B, \tau)$ is equal to number of points x_k in the set B and $N = N(Q, \tau) < \infty$ for fixed $\tau > 0$.

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Note, the case when $N(\cdot,\tau)=N(\cdot)$ there is random Poisson measure on a circle $C=\{(x,y):\quad x^2+y^2\le r^2\}$ was considered in [1].

There is exact formula of distribution function for number of remaining processes in this article. However, this formula consists diffucult computed functions. The present article is devoted to case when initial number of processes $N(Q,\tau)$ depends on the final time τ: $N(Q,\tau)\to\infty$ when $\tau\to\infty$. We shall obtain conditions of such dependence which leads to simple limit distribution function of number not absorbed processes.

Suppose that the region Q is bounded and boundary ∂Q is Lyapunov surface $C^{(1,\lambda)}$ [2]. We will consider the following case

$$a(t,x)=(\underbrace{0,\dots,0}_{d}),\quad b_i(t,x)=b_i=(b_{i1},\dots,b_{id}),\ 1\le i\le d.$$

We will define matrix $\sigma=B^T B,\quad B=(b_{ij}),\ 1\le i,j\le d\quad \sigma=(\sigma_{ij}),1\le i,j\le d$ and differential operator $A:\ \sum\limits_{1\le i,j\le d}\sigma_{ij}\frac{\partial^2}{\partial x_i\partial x_j}$. Let σ be a matrix with the following property

$$\sum_{1\le i,j\le d}\sigma_{ij}z_i z_j\ge\mu|\vec z|^2.$$

Here μ, there is fixed positive number, and $\vec z=(z_1,\cdots,z_d)$ there is an arbitrary real vector.

This operator acts in the following space

$$H_A=\{u:u\in L_2(Q)\cap Au\in L_2(Q)\cap u(\partial Q)=0\}$$

with inner product $(u,v)_A=(Au,v)$. Here $(,)$ is inner product in $L_2(Q)$. The operator A is positive operator [2]. It is known that the following eigenvalues problem

$$Au=-\lambda u,\quad u(\partial Q)=0$$

has infinity set of real eigenvalues $\lambda_i\to\infty$ and

$$0<\lambda_1<\lambda_2<\cdots<\lambda_s<\cdots.$$

The corresponding eigenfunctions

$$f_{11},\dots,f_{1n_1},\cdots,f_{s1},\dots,f_{sn_s},\cdots$$

form complete system of functions both in H_A and $L_2^0(Q):=\{u:u\in L_2(Q)\cap u(\partial Q)=0\}$. Here number n_k is equal to multiplicity of eigenvalue λ_k.

We will denote by $\eta(\tau)$ the number of remaining processes in the region D at the moment τ.

We will also assume that σ-additive measure ν is given on the Σ_ν- algebra sets from $Q,\quad \nu(Q)<\infty$. All eigenfunctions $f_{ij}:Q\to R^1$ are (Σ_ν,Σ_Y) measurable. Here Σ_Y is system of Borel sets from R^1. Let $\Rightarrow$ denote the weak convergence of random values or measures.

Let us denote by $\nu_\tau(\cdot)$ the measure

$$\nu_\tau(B)=\exp(-\frac{\tau}{2}\lambda_1)N(B,\tau).$$

where $B\in\Sigma_\nu$.

By definition of measure $\nu_\tau(\cdot)$, we have

$$d\nu_\tau(x)=\begin{cases}\exp(-\frac{\tau}{2}\lambda_1),&\text{if}\quad x=x_k,\quad k=1,\cdots,N(Q,\tau)\\0,&\text{otherwise.}\end{cases}$$

Theorem 1. *Suppose $N(\cdot, \tau)$ satisfies the condition*

$$\nu_\tau(\cdot) \underset{\tau \to \infty}{\Rightarrow} \nu(\cdot).$$

Then $\eta(\tau) \Rightarrow \eta$ if $\tau \to \infty$ where η has Poisson distribution function with parameter
$a = \int\limits_Q F(x)d\nu(x)$ *and* $F(x) = \sum\limits_{i=1}^{n_1} f_{1i}(x)c_{1i}, \quad c_{1i} = \int\limits_Q f_{1i}(x)dx.$

Proof. Consider the following initial-boundary problem

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{1 \le i,j \le d} \sigma_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad x \in Q;$$

$$u(0, x) = 1, \quad x \in Q$$

$$u(t, x) = 0 \quad \text{if} \quad x \in \partial Q, \quad t \ge 0 \tag{2}$$

It is known [3, sec. VIII], that $u(\tau, x)$ is equal to probability to remain in the region Q at the moment τ for a diffusion process which was in a point x at the initial moment ($\xi(0) = x \in Q$).

Introduce indicators

$$\chi_k(\tau) = \begin{cases} 1, & \text{if} \quad k - \text{th particle belongs to} \quad Q \text{ at the moment} \quad \tau \\ 0, & \text{if} \quad k - \text{th particle absorbed by the moment} \quad \tau. \end{cases}$$

These indicators are mutually independence by assumption. Thus the following relations are correct

$$\eta(\tau) = \sum_{k=1}^{N(Q,\tau)} \chi_k(\tau),$$

$$Es^{\eta(\tau)} = \prod_{k=1}^{N(Q,\tau)} Es^{\chi_k(\tau)} = \prod_{k=1}^{N(Q,\tau)} (su(\tau, \gamma_k) + (1 - u(\tau, \gamma_k)));$$

$$\ln Es^{\eta(\tau)} = \sum_{k=1}^{N(Q,\tau)} \ln(1 - u(\tau, \gamma_k)(1 - s)).$$

Here $0 \le s \le 1$. As $0 \le u(\tau, \gamma_k) \le 1$ then we have the following inequality from the last

$$\left| \ln Es^{\eta(\tau)} - \sum_{k=1}^{N(C,\tau)} u(\tau, \gamma_k)(1 - s) \right| \le \alpha \sum_{k=1}^{N(C,\tau)} u^2(\tau, \gamma_k), \tag{3}$$

Here $\alpha < \infty$. Define the value of $u(\tau, \gamma_k)$.

We shall assume that system of functions $\{f_{ij}(x), i \ge 1, 1 \le j \le n_i\}$ is orthonormalized with respect to space $L_2^0(Q)$. The ordinary argumentaion (see, for example, [2, sec. 22]) leads to the definition of solution of problem (2) of the form

$$u(t,x) = \sum_{k=1}^{\infty} \exp(-\frac{t}{2}\lambda_k) \sum_{j=1}^{n_k} c_{kj} f_{kj}(x)$$

where coefficients c_{ij} are equal to coefficients of decomposition of initial value (unit) by system of functions f_{ij}: $c_{ij} = \int_Q f_{ij}(x)dx$. The Parseval - Steklov equality is true for these coefficients:

$$\sum_{k=1}^{\infty} \sum_{j=1}^{n_k} c_{kj}^2 = |Q|. \tag{4}$$

Further

$$\sum_{k=1}^{N(Q,\tau)} u(\tau, \gamma_k) = \int_Q u(\tau, x)d\nu_\tau(x) =$$

$$= \int_Q F(x)d\nu_\tau(x) + \int_Q s_\tau(x)d\nu_\tau(x), \tag{5}$$

here

$$s_\tau(x) := \sum_{k \geq 2} \exp\left(-\frac{\tau}{2}(\lambda_k - \lambda_1)\right) \sum_{j=1}^{n_k} c_{kj} f_{kj}(x).$$

As function $F(x)$ is continuous and bounded function on the $\bar{Q}$ then under the condition of the theorem, we obtain [4]

$$\int_Q F(x)d\nu_\tau(x) \xrightarrow[\tau \to \infty]{} \int_Q F(x)d\nu(x).$$

In order to estimate of $s_\tau(x)$, we will give the result from monorgaphy [5,Thm. 17.5.3] Consider the sums of eigenfunctions of the form

$$e(x, \lambda) = \sum_{\lambda_k \leq \lambda} \sum_{j=1}^{n_k} f_{kj}^2(x)$$

then

$$\sup_{x \in Q} \sqrt{e(x, \lambda)} \leq C\lambda^{\frac{d}{2}}.$$

The asymptotic of eigenvalues λ_k under $k \to \infty$ is defined by the following eniqualities [2, sec. 18]

$$c_1 k^{\frac{2}{d}} \leq \lambda_k \leq c_2 k^{\frac{2}{d}}, \quad \text{where} \quad c_1, \ c_2 = const.$$

The last and (4) and Caushy-Bunyakovskii inequality lead to the following convergence under $\tau \to \infty$

$$|s_\tau(x)| \leq \sum_{k \geq 2} \exp\left(-\frac{\tau}{2}(\lambda_k - \lambda_1)\right) \sqrt{\sum_{j=1}^{n_k} c_{kj}^2} \sqrt{\sum_{m=1}^{n_k} f_{kj}^2(x)} \leq$$

$$\leq C \sum_{k \geq 2} \lambda_k^{\frac{d}{2}} \exp(-\frac{\tau}{2}(\lambda_k - \lambda)) \sqrt{\sum_{j=1}^{n_k} c_{kj}^2} \leq$$

$$\leq C \left(\sum_{k \geq 2} \lambda_k^d \exp(-\tau(\lambda_k - \lambda_1))\right)^{\frac{1}{2}} \left(\sum_{k \geq 2} \sum_{j=1}^{n_k} c_{kj}^2\right)^{\frac{1}{2}} \to 0. \qquad (6)$$

Combining the condition of the theorem and (6), we conclude that second summand of right part of (5) converges to zero when $\tau \to \infty$.

Now we shall estimate of the right part of inequality (3).

By definition of measure ν_τ we have

$$R_\tau := \sum_{k=1}^{N(Q,\tau)} u^2(\tau, \gamma_k) = \int_Q [J_1(\tau, x) + J_2(\tau, x) + J_3(\tau, x)] \, d\nu_\tau(x).$$

Here

$$J_1(\tau, x) := \exp(-\frac{\tau}{2}\lambda_1) \left(\sum_{j=1}^{n_1} c_{1j} f_{1j}(x)\right)^2;$$

$$J_2(\tau, x) := 2 \sum_{j=1}^{n_1} c_{1j} f_{1j}(x) \sum_{k \geq 2} \exp(-\frac{\tau}{2}\lambda_k) \sum_{j=1}^{n_k} c_{kj} f_{kj}(x);$$

$$J_3(\tau, x) := \exp(-\frac{\tau}{2}\lambda_1) s_\tau^2(x).$$

Observe that we have $F(x) \geq 0, x \in \bar{Q}$, because the function $u(t, x) \geq 0$ for all $t \geq 0$, $x \in \bar{Q}$.

Let $M = \max_{x \in \bar{Q}} F(x)$. Applying (6), we get under $\tau \to \infty$

$$|J_1(\tau, x)| \leq \exp(-\frac{\tau}{2}\lambda_1) M^2 \to 0;$$

$$|J_2(\tau, x)| \leq 2M \exp(-\frac{\tau}{2}\lambda_1) \sup_{x \in Q} |s_\tau(x)| \to 0;$$

$$|J_3(\tau, x)| \leq \exp(-\frac{\tau}{2}\lambda_1) \sup_{x \in Q} s_\tau^2(x) \to 0.$$

These inequalities and theorem's condition guarantee the convergence of R_τ to zero under $\tau \to \infty$.

The proof of theorem is complete.

Example. Now we shall investigate particular case of the general problem. Here we can calculate relevant values of normalizing function and on the other hand this case may be represent the first approximation of real situation.

Consider circle domain Q in E^2: $x^2+y^2 \leq r^2$. Assume that the diffusion particles start from point $(x_k, y_k) \in Q$ at the moment $t = 0$. The movement of particles is described by the following stochastic differential equations

$$d\xi(t) = \sum_i^2 b_i dw_i(t) \tag{7}$$

$$\xi(0) = \xi_0 = (x_k, y_k).$$

where $b_1 = (\sigma, 0), b_2 = (0, \sigma)$ and $W(t) = (w_i(t), i = 1, 2)$ be a 2-dimensional Wiener process.

Assume that the equation (7) defines a diffusion process with absorption on the boundary $\partial Q = \{(x, y) : x^2 + y^2 = r^2\}$.

In what follows, the $J_0(x), J_1(x)$ are Bessel functions zero and first order. It are defined as the solutions of next equations

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{n^2}{x^2}) = 0,$$

$$y(x_0) = 0, \ (x_0 = \sqrt{\lambda}r); \quad |y(0)| < \infty;$$

under $n = 0$ and $n = 1$.

The value of $\mu_m^{(0)}$ is equal to m- th root of equation $J_0(\mu) = 0$ [6,7].

We will use the symbol $mes(\cdot)$ to denote the Lebesgue measure.

Put

$$f(\tau) := \exp\left(-\frac{\tau}{2}\left(\frac{\sigma\mu_1^{(0)}}{r}\right)^2\right).$$

Suppose $N(\cdot, \tau)$ satisfies the condition

$$\lim_{\tau \to \infty} N(B, \tau)f(\tau) = mes(B), \quad B \in \Sigma_{mes}(C).$$

The initial position of k-th particle in this case has form $\gamma_k = (x_k, y_k)$

The value of $u(\tau, \gamma_k)$ is defined as value of $u(t, x, y)$ in this point. Here $u(t, x, y)$ be a solution of the following initial-boundary problem

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)u \quad (x, y) \in Q, \quad t > 0;$$

$$u(0, x, y) = 1 \quad (x, y) \in Q;$$

$$u(t, x, y) = 0 \quad \text{if} \quad (x, y) \in \partial D, \quad t \geq 0 \tag{8}$$

According to results from book [6, sec. VI] the solution of (8) in the point (τ, γ_k) can be expressed in form

$$u(\tau, \gamma_k) = \sum_{m=1}^{\infty} \frac{2J_0\left(\mu_m^{(0)}\frac{\sqrt{x_k^2+y_k^2}}{r}\right)}{\mu_m^{(0)} J_1\left(\mu_m^{(0)}\right)} \exp\left(-\frac{\tau}{2}\left(\frac{\sigma\mu_m^{(0)}}{r}\right)^2\right).$$

Let us compute the parameter a. In this case it is convenient to decompose of circle C by concentric circles for constuction of integal sums of integral $\int\limits_{C} F(x)\nu(dx)$ [8].

Define this partition

$$K_{ni} = \left\{ (x,y) \in C : \frac{ri}{n} \leq \sqrt{x^2 + y^2} < \frac{r(i+1)}{n} \right\}, \quad 0 \leq i \leq n-1.$$

Now $mes(K_{ni}) = g(\frac{i+1}{n}) - g(\frac{i}{n})$ where $g(\rho) = \pi r^2 \rho^2$, $0 \leq \rho \leq 1$.
Finally, the parameter of Poisson distribution is equal to

$$a = 2 \left(\mu_1^{(0)} J_1(\mu_1^{(0)}) \right)^{-1} 2\pi r^2 \int\limits_{0}^{1} J_0(\mu_1^{(0)}\rho)\rho d\rho = \pi \left(\frac{2r}{\mu_1^{(0)}} \right)^2 .$$

We used the following known relation $\alpha J_0(\alpha) = [\alpha J_1(\alpha)]'$ [6, sec.VI] for calculation of the last integral.

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